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RESEARCH PAPER

VISCOELASTIC FLOWS WITH FRACTIONAL DERIVATIVE MODELS: COMPUTATIONAL APPROACH BY CONVOLUTIONAL CALCULUS OF DIMOVSKI

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Dedicated to Professor Ivan Dimovski
on the occasion of his 80th anniversary

Abstract

An initial-boundary value problem for the velocity distribution of a viscoelastic flow with generalized fractional Oldroyd-B constitutive model is studied. The model contains two Riemann-Liouville fractional derivatives in time. The eigenfunction expansion of the solution is constructed. The behavior of the time-dependent components of the solution is studied and the results are used to establish convergence of the series under some conditions. Further, applying the convolutional calculus approach proposed by Dimovski (I.H. Dimovski, *Convolutional Calculus*, Kluwer, Dordrecht (1990)), a Duhamel-type representation of the solution is found, containing two convolution products of particular solutions and the given initial and source functions. A non-classical convolution with respect to spatial variable is used. The obtained representation is applied for numerical computation of the solution in the case of a generalized second grade fluid. Numerical results for several one-dimensional examples are given and the present technique is compared to a finite difference method in terms of efficiency, accuracy, and CPU time.

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1. Introduction

Linear viscoelasticity is certainly the field of the most extensive applications of Fractional Calculus, see e.g. [15, 16, 18, 19] and the references therein. This is due to the nonlocal character of fractional derivatives, leading to their ability to model more adequately phenomena with memory. On the other hand, the same nonlocality property makes it difficult to design fast and accurate numerical techniques for fractional order differential equations. Since many industrial and natural processes can be modeled as viscoelastic flows: from polymer extrusion to processes in geophysics, such numerical algorithms are essential.

Viscoelastic flows are intensively modeled in literature under different constitutive equations and in various media [8, 9, 12, 13, 24]. In this paper, we consider an initial-boundary value problem for the velocity distribution of a viscoelastic flow with generalized fractional Oldroyd-B constitutive model, see [8, 13, 24] for the derivation details.

Let Ω be a bounded rectangular domain in $\mathbb{R}^d$, $d = 1, 2$, with boundary $\partial\Omega$. Let $T > 0$ be a fixed time, and $0 < \alpha < \beta < 1$, $a, b \geq 0$, $\mu > 0$, be given constant parameters. Applying the generalized fractional Oldroyd-B constitutive model leads to the following initial-boundary value problem for the flow velocity $u(x, t)$:

$$\begin{aligned} &(1 + aD_t^{\alpha})u_t = \mu(1 + bD_t^{\beta})\Delta u + F(x,t), \quad x \in \Omega,\ t \in (0,T], \\ &u(x,t) = 0, \quad x \in \partial\Omega,\ t \in [0,T], \\ &u(x,0) = f(x), \quad x \in \overline{\Omega}, \end{aligned} \tag{1.1}$$

where Δ is the Laplacian acting on spatial variables, $u_t = \partial u/\partial t$ and D_t^{γ} is the Riemann-Liouville fractional derivative of order γ [10, 19]:

$$D_t^{\gamma} f(t) := \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_0^t \frac{f(\tau)}{(t-\tau)^{\gamma}}\, d\tau, \quad 0 < \gamma < 1.$$

If $a \neq 0$ an additional initial condition $u_t(x, 0) < \infty$ is assumed.

The generalized Oldroyd-B model (1.1) encompasses a large class of fluids [8, 24]: Newtonian fluid ($a = b = 0$), generalized second grade fluid ($a = 0$, $b > 0$), fractional Maxwell model ($b = 0$, $a > 0$).

Exact solutions for different versions of problem (1.1) in one and two spatial dimensions are obtained in the form of eigenfunction expansions, see [8, 11, 12, 13, 24]. However, the solutions given in these studies are formal in nature and the convergence of the series and the regularity of the solutions are not discussed.

Numerical algorithms for problem (1.1) with $a = 0$ (generalized second grade fluid) are studied extensively in the literature, see [3, 4, 5, 14, 17, 23]. To the best of our knowledge, there are no published numerical studies concerning the general case $a \neq 0$. In [4] and [5] implicit and explicit finite difference schemes have been examined for numerical solution of one- and two-dimensional problems, and Fourier analysis has been used to analyze the stability and convergence of the methods. In [23] an implicit numerical approximation scheme is developed by transforming the problem into an integral equation, in [14] a method based on the reproducing kernel is described, and in [17] a compact finite difference method and a radial basis function method are studied. In [3], the Sobolev regularity of the solution is established for both smooth and nonsmooth initial data and the regularity estimates are used to derive optimal with respect to data regularity error estimates for the developed there numerical approximations: a space semidiscrete Galerkin scheme using piecewise linear finite elements and two fully discrete approximations.

The convolutional calculus approach of Dimovski [6] is useful for obtaining compact Duhamel-type representations of the solutions of initial-boundary value problems, including multidimensional problems [7]. It is based on nonclassical convolutions with respect to spatial variables. This approach is applicable to problems with classical boundary conditions, as well as nonlocal boundary conditions, see e.g. [1, 2]. For a detailed recent study we refer also to [22].

In this paper, eigenfunction expansion of the solution of problem (1.1) is constructed, the behavior of the time-dependent components is studied and the convergence of the series under some conditions is established. Based on the convolutional calculus approach of Dimovski, a Duhamel-type representation of the solution is found, containing two convolution products of particular solutions and the given initial and source functions. This representation is used for numerical computation of the solution in the case of a generalized second grade fluid. The particular solutions are calculated using their eigenfunction expansions. Numerical results for several one-dimensional examples are given and the developed technique is compared to a finite difference method.

2. Preliminaries

Denote by $\overset{t}{*}$ the classical convolution:

$$(f \overset{t}{*} g)(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau,$$

which we use with respect to time variable (with respect to spatial variables we use nonclassical convolutions, defined in Section **5**). Let

$$\omega_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \ \ \alpha > 0. \tag{2.1}$$

Denote the Laplace transform of a function $f(t)$ by $\widehat{f}$ ot $\mathcal{L}\{f\}$. Then

$$\mathcal{L}\{\omega_\alpha\}(s) = s^{-\alpha}, \ \ \alpha > 0.$$

The Laplace transform for the Riemann-Liouville fractional differential operator D_t^α with $0 < \alpha < 1$ is given by [10]:

$$\mathcal{L}\{D_t^\alpha f\}(s) = s^\alpha \widehat{f}(s) - \left(\omega_{1-\alpha} \overset{t}{*} f\right)(0^+),$$

and thus (see [15], Ch. 1, eq. (1.29))

$$\mathcal{L}\{D_t^\alpha f\}(s) = s^\alpha \widehat{f}(s), \ \ \text{if} \ \ f(0) < \infty. \tag{2.2}$$

Denote by $E_{\alpha,\beta}(z)$ the Mittag-Leffler function [10, 19]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \ \ \alpha > 0, \ \beta \in \mathbb{R}, \ z \in \mathbb{C}.$$

The following identity is satisfied [10, 19]:

$$\mathcal{L}\{t^{\beta-1} E_{\alpha,\beta}(-\lambda t^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha + \lambda}, \ \ t > 0, \ \lambda \in \mathbb{R}. \tag{2.3}$$

Lemma **2.1**. *Let $a \neq 0$, $\alpha \in (0,1)$, $t \in (0,T]$. The ordinary fractional differential equation*

$$(1 + aD_t^\alpha) y(t) = f(t), \ y(0) < \infty,$$

has a unique solution given by

$$y(t) = h_\alpha(a,t) \overset{t}{*} f(t),$$

where

$$h_\alpha(a,t) := a^{-1} t^{\alpha-1} E_{\alpha,\alpha}(-a^{-1} t^\alpha). \tag{2.4}$$

P r o o f. The assertion follows applying Laplace transform to the equation and using properties (2.2) and (2.3). □

Note that, by applying Lemma **2.1** to the governing equation in (1.1) and then integrating both sides (or applying only the second step if $a = 0$), we can recast problem (1.1) into a Volterra integral equation:

$$u(x,t) = f(x) + \int_0^t k(t-\tau) \Delta u(x,\tau)\, d\tau + F_1(x,t),$$

where the kernel $k(t)$ is given by

$$k(t) = \begin{cases} \mu\left(1 + b\omega_{1-\beta}(t)\right), & \text{if } a = 0, \\ \mu h_\alpha(a,t) \overset{t}{*} \left(1 + b\omega_{1-\beta}(t)\right), & \text{if } a \neq 0, \end{cases}$$

and the function $F_1(x,t)$ is

$$F_1(x,t) = \begin{cases} \displaystyle\int_0^t F(x,\tau)\, d\tau, & \text{if } a = 0, \\ \displaystyle\int_0^t h_\alpha(a,\tau) \overset{\tau}{*} F(x,\tau)\, d\tau, & \text{if } a \neq 0. \end{cases}$$

Therefore, for the theoretical study of problem (1.1) we can apply the approach for abstract Volterra integral equations developed in the book [20]. In this paper we prefer to work using the eigenfunction expansion of the solution, but some ideas of [20] are used in Theorem **3.1**, where we establish the properties of the time-dependent components in the expansion.

3. Eigenfunction expansion of the solution and properties of the time-dependent components

Let $\{\lambda_n\}_{n\in\mathbb{N}}$ and $\{\varphi_n(x)\}_{n\in\mathbb{N}}$ be the Dirichlet eigenvalues and eigenfunctions of $-\Delta$ on the domain Ω, and let $0 < \lambda_1 < \lambda_2 < ....$ Denote by $(.,.)$ the inner product in $L^2(\Omega)$. Applying eigenfunction decomposition, the solution has the form

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t)\varphi_n(x),$$

where the functions $u_n(t)$ satisfy the following ordinary differential equation

$$\begin{aligned} &(1 + aD_t^\alpha)u_n'(t) = -\lambda_n\mu(1 + bD_t^\beta)u_n(t) + F_n(t), \\ &u_n(0) = f_n, \quad \text{if } a = 0, \\ &u_n(0) = f_n, \quad u_n'(0) < \infty, \quad \text{if } a \neq 0, \end{aligned} \tag{3.1}$$

where $f_n = (f, \varphi_n)$, $F_n(t) = (F(.,t), \varphi_n)$. We solve this problem by applying Laplace transform and obtain the formal eigenfunction expansion of the solution:

$$u(x,t) = \sum_{n=1}^{\infty} f_n G_n(t)\varphi_n(x) + \sum_{n=1}^{\infty} \left(\int_0^t H_n(t-\tau)F_n(\tau)\, d\tau\right)\varphi_n(x), \tag{3.2}$$

where the functions $G_n(t)$ and $H_n(t)$ are defined by their Laplace transforms as follows:

$$\widehat{G}_n(s) = \frac{1 + as^\alpha}{s(1 + as^\alpha) + \mu\lambda_n(1 + bs^\beta)}\,, \tag{3.3}$$

$$\widehat{H}_n(s) = \frac{1}{s(1 + as^\alpha) + \mu\lambda_n(1 + bs^\beta)}\,. \tag{3.4}$$

To prove that the series (3.2) is convergent we need estimates for the time-dependent components $G_n(t)$ and $H_n(t)$. Since in some aspects there are essential differences between the two cases $a = 0$ (parabolic equation) and $a \neq 0$ (nonparabolic), we consider them separately.

Denote by Σ_θ the sector

$$\Sigma_\theta := \{s \in \mathbb{C};\ s \neq 0, |\arg s| < \theta\}.$$

For $\rho > 0$ and $\theta \in (0, \pi)$ denote by $\Gamma_{\rho,\theta}$ the contour

$$\Gamma_{\rho,\theta} := \left\{re^{-\mathrm{i}\theta} :\ r \geq \rho\right\} \cup \left\{\rho e^{i\psi} :\ |\psi| \leq \theta\right\} \cup \left\{re^{\mathrm{i}\theta} :\ r \geq \rho\right\},$$

which is oriented counterclockwise.

Lemma **3.1**. *Let $a \neq 0$, $\rho > 0$, $\varphi_0 = \pi/(\alpha + 1)$ and fix $\varphi \in (\pi/2, \varphi_0)$. Then for any $n \in \mathbb{N}$ the function $\widehat{G}_n(s)$ has no poles in the sector $\overline{\Sigma}_{\varphi_0}$ and there hold the estimates*

$$\left|\widehat{G}_n(s)\right| \leq C|s|^{-1}, \quad \left|\lambda_n\widehat{G}_n(s)\right| \leq C\left(|s|^{-\beta} + a|s|^{\alpha-\beta}\right), \quad s \in \Sigma_\varphi. \tag{3.5}$$

P r o o f. Let $s_0 = re^{\mathrm{i}\theta}$, $r > 0$, $0 < \theta \leq \varphi_0$. Denote the denominator of $\widehat{G}_n(s)$ by $d_n(s)$. Then

$$\Im(d_n(s_0)) = r\sin\theta + ar^{\alpha+1}\sin(\alpha + 1)\theta + \mu\lambda_n br^\beta \sin\beta\theta > 0,$$

and thus s_0 can not be a zero of $d_n(s)$. This means that all poles of $\widehat{G}_n(s)$ lie outside the sector $\overline{\Sigma}_{\varphi_0}$.

For $s \in \Sigma_\varphi$ define the function

$$g(s) := \frac{s(1 + as^\alpha)}{\mu(1 + bs^\beta)}\,. \tag{3.6}$$

From (3.3) and (3.6) we get the representations:

$$\widehat{G}_n(s) = \frac{g(s)}{s(g(s) + \lambda_n)}, \quad \lambda_n\widehat{G}_n(s) = g(s)\left(\frac{1}{s} - \widehat{G}_n(s)\right). \tag{3.7}$$

We prove that if $s \in \Sigma_\varphi$ then $g(s) \in \Sigma_{\pi-\delta}$ for some $\delta \in (0, \pi/2)$. Let $s = re^{\mathrm{i}\psi}$, $|\psi| < \varphi$, $r > 0$. Then

$$g(s) = \frac{re^{\mathrm{i}\psi}(1 + ar^\alpha e^{\mathrm{i}\alpha\psi})}{\mu(1 + br^\beta e^{\mathrm{i}\beta\psi})} = \frac{re^{\mathrm{i}\psi}(1 + ar^\alpha e^{\mathrm{i}\alpha\psi})(1 + br^\beta e^{-\mathrm{i}\beta\psi})}{\mu(1 + br^\beta e^{\mathrm{i}\beta\psi})(1 + br^\beta e^{-\mathrm{i}\beta\psi})}$$
$$= \frac{1}{\mu} \frac{re^{\mathrm{i}\psi} + ar^{\alpha+1}e^{\mathrm{i}(\alpha+1)\psi} + br^{\beta+1}e^{\mathrm{i}(1-\beta)\psi} + abr^{\alpha+\beta+1}e^{\mathrm{i}(\alpha+1-\beta)\psi}}{(1 + br^\beta \cos(\beta\psi))^2 + (br^\beta \sin(\beta\psi))^2},$$

and by noting $0 < \alpha < \beta < 1$ we obtain $\arg(g(s)) \leq (\alpha + 1)\psi$. Hence $g(s) \in \Sigma_{\pi-\delta}$ with $\delta = \pi - (\alpha + 1)\varphi$, i.e. $\delta \in (0, \pi/2)$. Based on this, we prove that for any real constant $c > 0$

$$\left| \frac{g(s)}{g(s) + c} \right| \leq C, \quad s \in \Sigma_\varphi, \tag{3.8}$$

which together with the first identity in (3.7) gives the first estimate in (3.5). Indeed, the elementary inequality $x^2 + 2ax + 1 \geq 1 - a^2$ for any $x, a \in \mathbb{R}$ implies

$$1 + 2x\cos\theta + x^2 \geq \sin^2\theta, \quad x \in \mathbb{R}. \tag{3.9}$$

Since $g(s) \in \Sigma_{\pi-\delta}$, i.e. $g(s) = re^{\mathrm{i}\theta}$, $r > 0$, $|\theta| < \pi - \delta$, using (3.9) it follows

$$\left| \frac{g(s)}{g(s) + c} \right|^2 = \frac{r^2}{r^2 + 2cr\cos\theta + c^2} \leq \frac{1}{\sin^2\theta},$$

which implies (3.8) with $C = (\sin\delta)^{-1}$.

Further, the first estimate in (3.5) together with (3.7) gives

$$\left| \lambda_n \widehat{G}_n(s) \right| \leq C \left| \frac{g(s)}{s} \right| = C \left| \frac{1 + as^\alpha}{\mu(1 + bs^\beta)} \right|. \tag{3.10}$$

Since for $s = re^{\mathrm{i}\psi}$, $\psi \in (\pi/2, \pi)$

$$|1 + bs^\beta| = \Big((1 + br^\beta \cos(\beta\psi))^2 + (br^\beta \sin(\beta\psi))^2\Big)^{1/2} \geq br^\beta \sin(\beta\pi),$$

inserting this estimate in (3.10) gives the second estimate in (3.5). This completes the proof of the lemma. $\square$

THEOREM **3.1**. *Let $a \neq 0$. The functions $G_n(t)$ and $H_n(t)$, $n \in \mathbb{N}$, are continuous for $t \geq 0$ and have the following properties:*

$$G_n(0) = 1, \ H_n(0) = 0, \tag{3.11}$$

$$|G_n(t)| \leq C, \ t \geq 0, \tag{3.12}$$

$$|\lambda_n G_n(t)| \leq C\left(t^{\beta-1} + a t^{\beta-\alpha-1}\right), \quad t > 0, \tag{3.13}$$

$$H_n(t) = h_\alpha(a,t) \overset{t}{*} G_n(t), \ t \geq 0, \tag{3.14}$$

$$\int_0^t |\lambda_n H_n(\tau)| \, d\tau \leq C, \ t \geq 0, \tag{3.15}$$

where the function $h_\alpha(a,t)$ is defined in (2.4) and the constants C do not depend on n and t.

P r o o f. Applying the property of the Laplace transform

$$f(0) = \lim_{s \to +\infty} s\widehat{f}(s)$$

to (3.3) and (3.4), we obtain (3.11).

Further, taking the inverse Laplace transform of (3.3), we get

$$G_n(t) = \frac{1}{2\pi i} \int_{Br} e^{st} \frac{1 + as^\alpha}{s(1 + as^\alpha) + \mu\lambda_n(1 + bs^\beta)} \, ds, \tag{3.16}$$

with Br the Bromwich path: $Br = \{s; \ \Re s = \sigma\}$, where $\sigma > 0$ and $\sigma > \Re s_j$, s_j being the singularities of the Laplace transform. According to Lemma **3.1** the function under the integral has no poles in the sector $\overline{\Sigma}_{\varphi_0}$, where $\varphi_0 = \pi/(\alpha+1)$. Therefore we can bend the Bromwich path in (3.16) into the contour

$$\Gamma := \Gamma_{1/t,\varphi}, \quad t > 0, \ \varphi \in (\pi/2, \varphi_0),$$

and prove (3.12) by applying the first estimate in (3.5) (note that $\cos\varphi < 0$):

$$\begin{aligned} |G_n(t)| &\leq C \int_\Gamma e^{\Re(s)t} |s|^{-1} \, |ds| \\ &\leq C\left(\int_{1/t}^\infty e^{rt\cos\varphi} r^{-1} \, dr + \int_0^\varphi e^{\cos\psi} \, d\psi\right) \leq C. \end{aligned}$$

Further, applying the second estimate in (3.5) we get

$$|\lambda_n G_n(t)| \leq C \int_\Gamma e^{\Re(s)t} \left(|s|^{-\beta} + a|s|^{\alpha-\beta}\right) |ds|,$$

and noting that for $\gamma \in (0,1)$

$$\int_\Gamma e^{\Re(s)t}|s|^{-\gamma}\,|ds| \le C\left(\int_{1/t}^\infty e^{rt\cos\varphi}r^{-\gamma}\,dr + \int_0^\varphi e^{\cos\psi}t^{\gamma-1}\,d\psi\right) \le Ct^{\gamma-1},$$

we obtain (3.13).

Based on the Laplace transforms of $G_n(t)$ and $H_n(t)$, (3.3) and (3.4), and the property (2.2) it follows that

$$(1 + aD_t^\alpha)H_n(t) = G_n(t).$$

Then, since $H_n(0) = 0$, Lemma **2.1** implies (3.14).

Now, from (3.14) and the Young inequality for the classical convolution we get

$$\int_0^t |\lambda_n H_n(\tau)|\,d\tau \le \int_0^t |h_\alpha(a,\tau)|\,d\tau \int_0^t |\lambda_n G_n(\tau)|\,d\tau$$

and, inserting (3.13) and the inequality $|h_\alpha(a,t)| \le Ct^{\alpha-1}$, which is implied by the boundedness of the Mittag-Leffler function, we establish the last estimate (3.15). □

For the case $a = 0$, properties of the time-dependent components $G_n(t)$ and $H_n(t)$ (in this case $G_n(t) = H_n(t)$), are summarized in [3], Theorem 2.2. This theorem implies that estimates (3.12), (3.13) and (3.15) hold for $a = 0$. In addition, the following result is proven:

THEOREM **3.2**. *Assume $a = 0$. Then the functions $G_n(t)$ and $H_n(t)$ in the eigenfunction expansion (3.2) have the representation*

$$G_n(t) = H_n(t) = \int_0^\infty e^{-rt}K_n(r)\,dr, \quad n \in \mathbb{N}, \tag{3.17}$$

where

$$K_n(r) = \frac{b}{\pi}\frac{\mu\lambda_n r^\beta \sin\beta\pi}{(-r + \mu\lambda_n b r^\beta\cos\beta\pi + \mu\lambda_n)^2 + (\mu\lambda_n b r^\beta \sin\beta\pi)^2}\,. \tag{3.18}$$

P r o o f. The function under the Laplace inverse integral (3.16) has a branch point 0, so we cut off the negative part of the real axis. When $a = 0$ this function has no poles in the main sheet of the Riemann surface including its boundaries on the cut, since

$$\Im(s + \mu\lambda_n(1 + bs^\beta)) \ne 0.$$

So, we can bend the Bromwich path into the Hankel path, which starts from $-\infty$ along the lower side of the negative real axis, encircles the origin counterclockwise and ends at $-\infty$ along the upper side of the negative real axis, and obtain from (3.16) the representation (3.17). More details are given in [3], Theorem 2.2. □

Based on the properties of the time-dependent components we prove that the obtained formal solution (3.2) of problem (1.1) is a continuous function under some conditions. For the sake of brevity we consider here only the $1D$ case. The proof in $2D$ case is analogous.

Consider the one-dimensional problem (1.1) on $\Delta = (0,1) \times (0,T)$:

$$
\begin{aligned}
&(1 + aD_t^{\alpha})u_t = \mu(1 + bD_t^{\beta})u_{xx} + F(x,t),\\
&u(0,t) = u(1,t) = 0, \qquad\qquad (3.19)\\
&u(x,0) = f(x), \quad u_t(x,0) < \infty.
\end{aligned}
$$

Then the corresponding eigenvalues and eigenfunctions are respectively $\lambda_n = n^2\pi^2$ and $\varphi_n(x) = \sqrt{2}\sin(n\pi x)$, $n \in \mathbb{N}$.

Theorem **3.3**. *Let $f(x) \in C^2([0,1])$, $f(0) = f(1) = 0$, $F(x,t) \in C(\overline{\Delta})$. Then the function $u(x,t)$ defined in (3.2) satisfies $u \in C(\overline{\Delta})$.*

P r o o f. First note that $\varphi_n(x)$ are bounded functions on $[0,1]$. Since $f(x) \in C^2([0,1])$ and $f(0) = f(1) = 0$, then after integration by parts in the identity $f_n = (f, \varphi_n)$ we get $|f_n| \le Cn^{-2}$. This together with the estimate (3.12) implies that the first series in (3.2) is uniformly convergent on $\overline{\Delta}$. Analogously, the uniform convergence of the second series in (3.2) is implied by the estimate (3.15) and the boundedness of $|F(x,t)|$ on $\overline{\Delta}$. Since all terms in the series are continuous functions on $\overline{\Delta}$, their sum is a continuous function on $\overline{\Delta}$. □

Remark **3.1**. In the homogeneous case $F \equiv 0$, with f satisfying assumptions of Theorem **3.3**, we can prove better regularity of the solution. Indeed, by differentiating termwise the series (3.2) and using similar argument as in the proof of Theorem **3.3**, together with estimates (3.13), it follows $u_{xx}(x,t) \in C(\Delta)$. To prove this in the inhomogeneous case, we would need additional assumptions on F.

Remark **3.2**. The obtained estimates for the functions $G_n(t)$ and $H_n(t)$ are useful for further study of solution regularity in different settings, e.g. in Sobolev spaces, as in [21] for the fractional diffusion equation.

4. Finite difference approximation

In this section we construct difference schemes for the one-dimensional problem (3.19). Assume $x \in [0,1], t \in [0,T]$. Let M and N be the number of time and space nodes, and $\tau = T/M$, $h = 1/N$ be the time and space steps, respectively. Let $x_j := jh,\ t_k := k\tau,\ j = 0,1,...,N, k = 0,1,...,M$. We approximate problem (3.19) by an implicit and an explicit finite difference schemes, based on the Grünwald-Letnikov approximation of the Riemann-Liouville fractional derivative [19]:

$$(D_t^\alpha u)_j^k = \tau^{-\alpha} \sum_{m=0}^{k} (-1)^m \binom{\alpha}{m} u_j^{k-m} + O(\tau), \tag{4.1}$$

and the usual approximations for for the integer order derivatives u_{xx} and u_t:

$$\delta_x^2 u_j^l = \frac{u_{j-1}^l - 2u_j^l + u_{j+1}^l}{h^2}, \quad \delta_t^+ u_j^l = \frac{u_j^{l+1} - u_j^l}{\tau}, \quad \delta_t^- u_j^l = \frac{u_j^l - u_j^{l-1}}{\tau}.$$

In this way we obtain

$$\delta_t u_j^k + \frac{a}{\tau^\alpha} \sum_{m=0}^{k} w_m^\alpha \delta_t u_j^{k-m} = \mu \left(\delta_x^2 u_j^k + \frac{b}{\tau^\beta} \sum_{m=0}^{k} w_m^\beta \delta_x^2 u_j^{k-m} \right), \tag{4.2}$$

where $w_m^\alpha = (-1)^m \binom{\alpha}{m}$ and $\delta_t u_j^l = \delta_t^+ u_j^l$ for the explicit scheme and $\delta_t u_j^l = \delta_t^- u_j^l$ for the implicit scheme. The initial and boundary conditions are discretized in a standard way (nonhomogeneous boundary conditions can also be considered).

In the case $a = 0$ the explicit and implicit schemes obtained from (4.2) are analyzed in [4]. It should be possible to prove convergence and stability results for the schemes (4.2) in the general case $a \neq 0$ by a method similar to that used in [4]. However, since this is out of the scope of the paper, we leave it as an open problem for future research. In [4] it is proven that for the explicit scheme the following stability condition is required

$$4\mu\tau(1 + b\tau^{-\beta}) \leq h^2. \tag{4.3}$$

In order to satisfy this condition, for reasonably fine mesh in space, we should take extremely small time steps if β increases. We considered some test problems and found out that difficulties in numerical implementation of the explicit scheme appear for $\beta \geq 0.3$. In addition, we encountered similar problems also for $a \neq 0$.

The implicit scheme is unconditionally stable. However, its numerical implementation is based on solving a system of algebraic equations for each time step, which again leads to problems when using fine meshes, especially for large times T or $2D$ problems.

The above mentioned difficulties motivate us to look also for nonstandard methods for numerical solution of problem (1.1).

5. Duhamel-type representation of the solution

Applying the convolutional calculus of Dimovski [6], in this section we obtain a Duhamel-type representation of the solution of the one-dimensional problem (3.19).

Basic in a convolutional calculus is the notion of convolution.

DEFINITION **5.1**. [6] Let $L : X \to X$ be a linear operator defined on a linear space X. A bilinear, commutative and associative operation $* : X \times X \to X$ is said to be a convolution of the operator L iff

$$L(f*g) = (Lf)*g \quad \text{for any } f, g \in X.$$

Following [6], we define in the space $C([0,1])$ of continuous functions on $[0,1]$ the operator L, which is right inverse of the operator $D = d^2/dx^2$ and satisfies $(Lf)(0) = (Lf)(1) = 0$. It is given explicitly by

$$Lf(x) = \int_0^x (x-\xi) f(\xi)\, d\xi - x \int_0^1 (1-\xi) f(\xi)\, d\xi.$$

The following operation is a convolution of the operator L, [6]:

$$\begin{aligned}(f \overset{x}{*} g)(x) = -\frac{1}{2}\int_0^1 \bigg(&\int_x^\xi f(\xi + x - \eta) g(\eta)\, d\eta \\ &- \int_{-x}^\xi f(|\xi - x - \eta|) g(|\eta|) \operatorname{sgn}((\xi - x - \eta)\eta)\, d\eta \bigg)\, d\xi.\end{aligned} \tag{5.1}$$

Moreover,

$$Lf = \{x\} \overset{x}{*} f. \tag{5.2}$$

The following properties hold true, [6]:

$$D(f \overset{x}{*} g) = (Df) \overset{x}{*} g + D((Ff) \overset{x}{*} g), \quad F(f \overset{x}{*} g) = F((Ff) \overset{x}{*} g), \tag{5.3}$$

where F is the defining projector, $F = I - LD$. It incorporates the boundary conditions and has the explicit form $Ff(x) = f(0)(1-x) + f(1)x$. Denote

$$f \overset{x}{\tilde{*}} g := D(f \overset{x}{*} g). \tag{5.4}$$

THEOREM **5.1**. *The operation*

$$(f \overset{x}{\widetilde{*}} g)(x) = -\frac{1}{2}\frac{d}{dx}\left(\int_x^1 f(1+x-\eta)g(\eta)\,d\eta \right. \\ \left. + \int_{-x}^1 f(|1-x-\eta|)g(|\eta|)\mathrm{sgn}((1-x-\eta)\eta)\,d\eta\right) \tag{5.5}$$

is a convolution of the operator L in $C^1([0,1])$ such that the representation $Lf = \{Lx\} \overset{x}{\widetilde{*}} f$ holds. Moreover, for $m, n \in \mathbb{N}$

$$\sin(n\pi x) \overset{x}{\widetilde{*}} \sin(m\pi x) = \begin{cases} 0, & m \neq n, \\ (-1)^{n-1}\dfrac{n\pi}{2}\sin(n\pi x), & m = n. \end{cases} \tag{5.6}$$

P r o o f. Expression (5.5) is obtained by differentiation of (5.1). Property (5.6) can be proven directly. The rest follows from the properties of the original convolution (5.1). Detailed proof is a matter of direct but tedious check, so it will be omitted. □

Based on the convolution $\overset{x}{\widetilde{*}}$ and on the classical convolution $\overset{t}{*}$, we define a bivariate convolution $\overset{x,t}{\widetilde{*}}$ of two functions $f(x,t)$ and $g(x,t)$:

$$(f \overset{x,t}{\widetilde{*}} g)(x,t) = \int_0^t f(x,t-\tau) \overset{x}{\widetilde{*}} g(x,\tau)\,d\tau. \tag{5.7}$$

The Duhamel-type representation of the solution is given in the next

THEOREM **5.2**. *If $f(x) \in C^2([0,1])$, $f(0) = f(1) = 0$ and $F_t(x,t) \in C(\overline{\Delta})$, then the solution of problem (3.19) has the following representation:*

$$u(x,t) = U \overset{x}{\widetilde{*}} f'' + \frac{\partial}{\partial t}(V \overset{x,t}{\widetilde{*}} F), \tag{5.8}$$

where $U(x,t)$ is a particular solutions of (3.19) with $f(x) = (x^3 - x)/6$ and $F \equiv 0$; $V(x,t)$ is a particular solutions of (3.19) with $f \equiv 0$, $F(x,t) = x$. The functions $U(x,t)$ and $V(x,t)$ have the following series expansions:

$$U(x,t) = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}G_n(t)\sin(n\pi x), \\ V(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\int_0^t H_n(\tau)\,d\tau\right)\sin(n\pi x). \tag{5.9}$$

P r o o f. Inserting the Fourier coefficients of the functions $(x^3 - x)/6$ and x in (3.2), we obtain the eigenfunction expansions (5.9). By the estimates (3.12) and (3.15) it is clear that the series in (5.9) are uniformly convergent, as well as the series after termwise differentiation with respect to x. Therefore U_x and V_x are continuous functions on $[0,1]\times[0,T]$ and the expression in (5.8) is well-defined. We prove that formula (5.8) is equivalent to (3.2). Indeed, inserting the eigenfunction expansions of U and V in (5.8), together with the eigenfunction expansions of arbitrary initial function $f(x)$ and source function $F(x,t)$, and using the properties

$$(f \overset{t}{*} g)' = f' \overset{t}{*} g + f(0)g(t), \tag{5.10}$$

(5.6) and the separability property of the bivariate convolution [7]

$$(f_1(x)f_2(t)) \overset{x,t}{\tilde{*}} (g_1(x)g_2(t)) = \left(f_1(x) \overset{x}{\tilde{*}} g_1(x)\right)\left(f_2(t) \overset{t}{*} g_2(t)\right),$$

we get (3.2). $\square$

Remark **5.1**. Note that the functions $\{(x^3-x)/6\}$ and $\{x\}$, appearing in Theorem **5.2**, play special role in our convolutional approach. Due to the representation (5.2) of the operator L as a convolution operator, the function $\{x\}$ can be identified with the operator L. Since

$$L\{x\} = \{(x^3 - x)/6\},$$

the function $\{(x^3 - x)/6\}$ can be identified with L^2. For more details we refer to [6] or [22].

Applying properties (5.7) and (5.10), we can differentiate the convolution product in (5.8) and obtain:

Corollary **5.1**. *Under the assumptions of Theorem* **5.2**, *we have*

$$u(x,t) = U \overset{x}{\tilde{*}} f''(x) + V \overset{x,t}{\tilde{*}} \left(\frac{\partial F}{\partial t}\right) + V \overset{x}{\tilde{*}} F(x,0). \tag{5.11}$$

6. Numerical experiments

For the numerical implementation of the Duhamel-type representations of the solution (5.8)/(5.11), the particular solutions $U(x,t)$ and $V(x,t)$ can be calculated in advance by an appropriate method. Then the solution $u(x,t)$ of (3.19) with arbitrary initial function $f(x)$ and source function $F(x,t)$ is computed according to (5.8)/(5.11) applying only numerical integration and differentiation.

As usual, the right-hand side $F(x,t)$ can incorporate nonhomogeneous boundary conditions. Note that such conditions are often considered with this problem, e.g. when flow between two moving parallel plates is modeled, e.g. [13]. Indeed, the solution of the problem with nonhomogeneous boundary conditions

$$\begin{aligned}
&(1+aD_t^\alpha)u_t = \mu(1+bD_t^\beta)u_{xx},\\
&u(0,t)=\phi(t),\ \ u(1,t)=\psi(t),\\
&u(x,0)=0,\ \ u_t(x,0)<\infty,
\end{aligned}$$

is given by $u = v + \phi(t)(1-x) + \psi(t)x$, where v is a solution of (3.19) with $f \equiv 0$ and

$$F(x,t) = -(1+aD_t^\alpha)(\phi'(t)(1-x) + \psi'(t)x).$$

For numerical implementations, the function $F(x,t)$ can be computed using the Grünwald-Letnikov approximation of the Riemann-Liouville fractional derivative (4.1).

In order to compare the proposed method for numerical computation of the solution to a standard finite difference method, and to visualize the solution in some practically interesting cases, we present in this section some numerical examples. We carried out numerical experiments only for the case $a = 0$ because of two reasons. First, as mentioned in Section **1**, the existing numerical studies for (1.1) are mostly concerned with this case. Second, for $a = 0$ it is easy to compute the kernels U and V, using their eigenfunction expansions, which makes our method self-contained.

Expansions (5.9) and Theorem **3.2** imply that the particular solutions of $1D$ problem (3.19) with $a = 0$ are given by:

$$U(x,t) = \frac{2}{\pi^3}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}G_n(t)\sin(n\pi x), \tag{6.1}$$

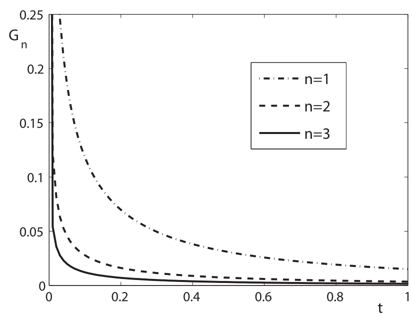
$$V(x,t) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}S_n(t)\sin(n\pi x), \tag{6.2}$$

where

$$G_n(t) = \int_0^\infty e^{-rt}K_n(r)\,dr, \quad S_n(t) = \int_0^\infty \frac{1-e^{-rt}}{r}K_n(r)\,dr,$$

with function $K_n(r)$ given in (3.18), where $\lambda_n = n^2\pi^2$. In the numerical tests in this section we use representations (6.1) and (6.2) for the numerical computation of $U(x,t)$ and $V(x,t)$.

To get an impression of the behavior of the time-dependent components in the series (6.1) and (6.2), on Fig. 1. we give some plots of $G_n(t)$ and $S_n(t)$.



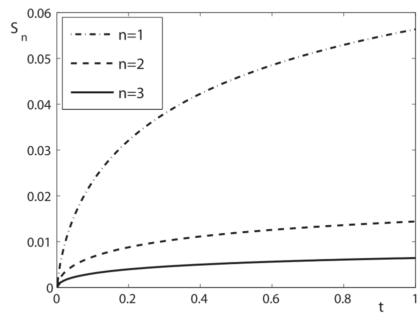


FIGURE 1. Plots of the functions $G_n(t)$ and $S_n(t)$, for $\beta = 0.5$, $\mu = b = 1$, $n = 1, 2, 3$.

First, consider a test problem with known exact solution in closed form and compare our method, based on formula (5.8), with the explicit finite difference method from [4]. We choose the explicit scheme because it is easy for numerical implementation, a feature which is also characteristic for our method.

EXAMPLE **6.1**. Consider the following problem

$$\begin{aligned} &u_t = (1 + D_t^{\beta})u_{xx} + F(x,t), \\ &u(0,t) = u(1,t) = u(x,0) = 0. \end{aligned} \tag{6.3}$$

To compute its solution based on representation (5.11), we compute first the kernel $V(x,t)$ using (6.2), see Fig. 2. (left). Recall that V is a solution of (6.3) with $F(x,t) = x$. On Fig. 2. (right) the solution of problem (6.3) is given for

$$F(x,t) = \big(2t + \pi^2 t^2 + 2\pi^2 \omega_{3-\beta}(t)\big)\sin(\pi x), \tag{6.4}$$

where $\omega_{3-\beta}(t)$ is defined by (2.1). For this special choice of F problem (6.3) has an exact solution: $u_{\text{exact}}(x,t) = t^2 \sin(\pi x)$. We make numerical tests to compare two methods for solving (6.3): the convolutional calculus method (CCM) (representation (5.11)) and the explicit finite difference method (FDM). We choose $\beta = 0.2$ because of the stability requirement (4.3) on the explicit scheme. Note that our method is applicable in the whole range $\beta \in (0,1)$.

On Fig. 3. results of the comparisons are presented. The errors calculated by the formula

$$\max_{0 \le k \le N} |u(x_k,t) - u_{\text{exact}}(x_k,t)|$$

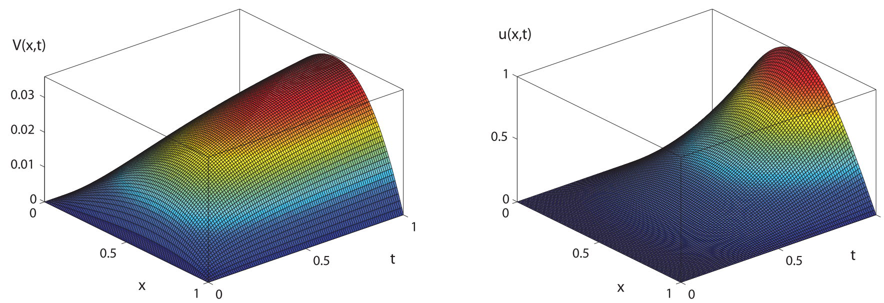


FIGURE 2. Two solutions of problem (6.3) with $\beta = 0.2$: the kernel $V(x,t)$ and solution with $F(x,t)$ given by (6.4).

are given and the times necessary to achieve the corresponding accuracy. It is seen that in order to achieve similar accuracy for this test problem, we need 9×10^3 more time if we use FDM, compared to CCM. (The time necessary for computation of the kernel $V(x,t)$ is not included, since this is usually done in advance. This time is less than 10 min.)

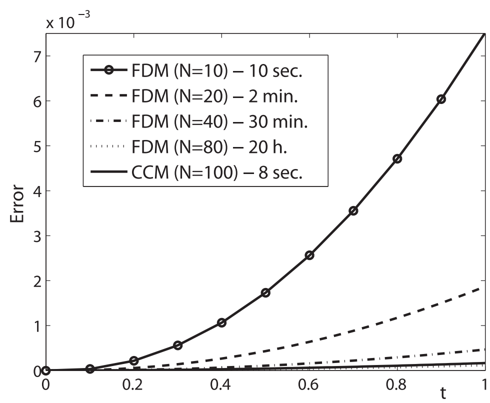


FIGURE 3. Comparisons of CCM with FDM for the test problem from Example **6.1** (N - number of space nodes).

EXAMPLE **6.2**. Consider the problem

$$\begin{aligned} &u_t = (1 + D_t^{\beta})u_{xx}, \\ &u(0,t) = u(1,t) = 0, \;\; u(x,0) = f(x). \end{aligned} \tag{6.5}$$

It models velocity distribution of a flow with nonzero initial velocity $f(x)$ and situated between two parallel plates at rest (with a no-slip condition). On Fig. 4. we give the graphs of two solutions of (6.5) with $\beta = 0.25$: the left is of the kernel $U(x,t)$ (corresponds to initial function $f(x) = (x^3 - x)/6$, computed using (6.1)), the right is with initial function $f(x) = \sin(2\pi x)$ and is computed based on representation (5.11). A comparison with the exact solution for this case $u_{\text{exact}}(x,t) = G_2(t)\sin(2\pi x)$, gives error of the order of 10^{-4} for $M = N = 100$. The relatively short time intervals on the figures are chosen because of the fast decay of the solution.

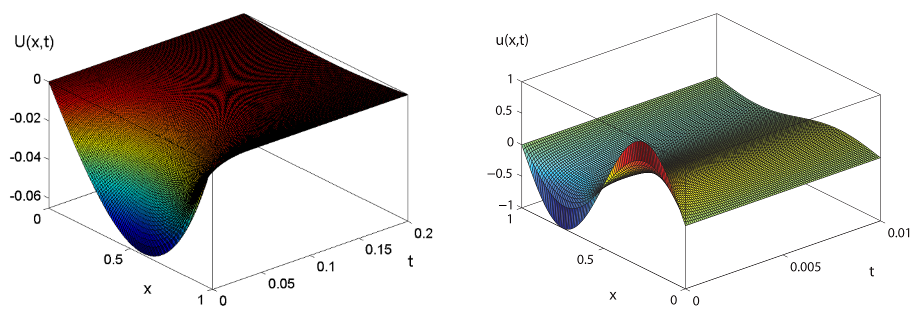


FIGURE 4. Two solutions of problem (6.5) with $\beta = 0.25$: the kernel $U(x,t)$ and the solution with $f(x) = \sin(2\pi x)$.

EXAMPLE **6.3**. The solutions of (6.5) with $f(x) = x(1-x)$ are computed for different values of β: $\beta = 0.25, 0.5$ and 0.75, using (5.11). On Fig. 5. the graph of the solution is given for $\beta = 0.25$ (left) and the decays of the solutions at $x = 0.5$ are compared for different values of β (right).

EXAMPLE **6.4**. Consider the problem

$$\begin{aligned} &u_t = (1 + D_t^\beta)u_{xx}, \\ &u(0,t) = \phi(t),\ u(1,t) = 0,\ u(x,0) = 0. \end{aligned} \tag{6.6}$$

It models velocity distribution of a flow between two parallel plates, one of which is moving. The flow is initially at rest. On Fig. 6. the graphs of the solution for $\beta = 0.5$ are given in two cases: the flow is induced by a linear acceleration ($\phi(t) = t^2$, right) or oscillation ($\phi(t) = \sin(4\pi t)$, left) of the moving plate, together with a no-slip condition. The solution has the form $u = v + \phi(t)(1-x)$, where v is the solution of problem (6.3) with $F(x,t) = -\phi'(t)(1-x)$. We calculate v using formula (5.11).

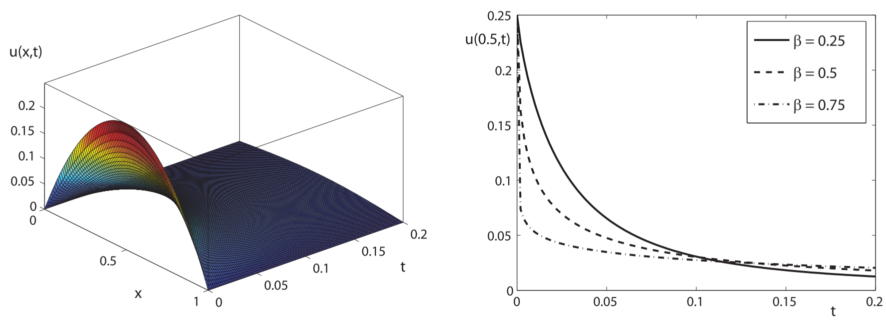


FIGURE 5. Plots of the solutions from Example **6.3**.

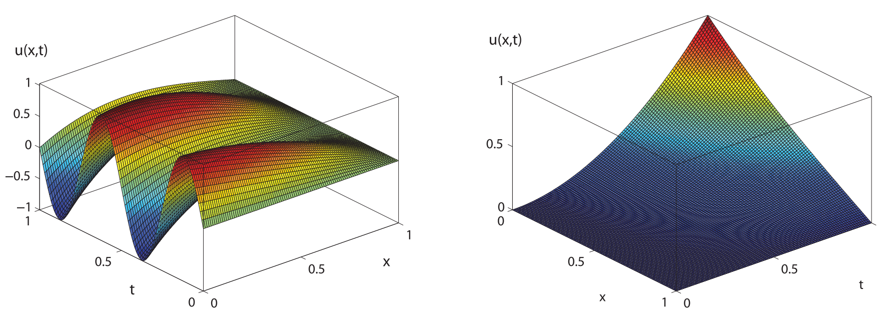


FIGURE 6. Flow between two parallel plates, one of which is oscillating (left) or moving with linear acceleration (right).

For the computation of the convolution products in the numerical tests of this section we have used use Simpson's rule for numerical integration $(O(h^4))$, and central differences for numerical differentiation $(O(h^2))$. So, the numerical approximation of the convolutions is not optimal and further improvement of the accuracy is possible. We take $T \leq 1$ ($T << 1$ when the solution has a fast decay). However, there are no obstacles in numerical implementation to to work with large times, since the solution is computed in each point independently. Also, there are no limitations on the order $\beta \in (0,1)$ of fractional differentiation.

7. Two-dimensional problem

In this section we briefly present the application of the convolutional approach to the $2D$ problem on the unit square $\Omega = [0,1]\times[0,1]$

$$\begin{aligned} &(1 + aD_t^{\alpha})u_t = \mu(1 + bD_t^{\beta})(u_{xx} + u_{yy}) + F(x,y,t), \\ &u(0,y,t) = u(1,y,t) = u(x,0,t) = u(x,1,t) = 0, \\ &u(x,y,0) = f(x,y), \quad u_t(x,y,0) < \infty. \end{aligned} \tag{7.1}$$

To find Duhamel-type representation of the solution, we apply the general scheme for constructing multidimensional convolutional calculi, see e.g. [7]. Define the bivariate convolution $\overset{x,y}{\widetilde{*}}$ of functions $f, g \in C(\Omega)$ as a composition of two convolutions (5.5):

$$(f \overset{x,y}{\widetilde{*}} g)(x,y) = -\frac{1}{4}\frac{\partial}{\partial x}\frac{\partial}{\partial y}K(x,y),$$

where

$$\begin{aligned} K(x,y) &= \int_x^1\int_y^1 f(1+x-\xi, 1+y-\eta)g(\xi,\eta)\,d\xi d\eta \\ &+ \int_x^1\int_{-y}^1 f(1+x-\xi, |1-y-\eta|)g(\xi,|\eta|)\operatorname{sgn}((1-y-\eta)\eta)\,d\xi d\eta \\ &+ \int_{-x}^1\int_y^1 f(|1-x-\xi|, 1+y-\eta)g(|\xi|,\eta)\operatorname{sgn}((1-x-\xi)\xi)\,d\xi d\eta \\ &+ \int_{-x}^1\int_{-y}^1 f(|1-x-\xi|, |1-y-\eta|)g(|\xi|,|\eta|) \\ &\qquad\qquad \times \operatorname{sgn}((1-x-\xi)(1-y-\eta)\xi\eta)\,d\xi d\eta. \end{aligned}$$

Define also the convolution

$$(F \overset{x,y,t}{\widetilde{*}} G)(x,y,t) = \int_0^t F(x,y,t-\tau) \overset{x,y}{\widetilde{*}} G(x,y,\tau)\,d\tau.$$

Theorem 7.1. *Let* $\dfrac{\partial^4 f(x,y)}{\partial x^2 \partial y^2} \in C(\Omega)$, $f(0,y) = f(1,y) = f(x,0) = f(x,1) = 0$ *and* $F_t(x,y,t) \in C(\Omega\times[0,T])$. *Then the solution of problem (7.1) has the representation*

$$u(x,y,t) = U \overset{x,y}{\widetilde{*}} \left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right) + \frac{\partial}{\partial t}\left(V \overset{x,y,t}{\widetilde{*}} F\right),$$

where $U(x,y,t)$ *is a particular solution of (7.1) with* $F \equiv 0$ *and* $f(x,y) = (x^3 - x)(y^3 - y)/36$; $V(x,y,t)$ *is a particular solution of (7.1) with* $f \equiv 0$ *and* $F(x,y,t) = xy$.

Corollary **7.1**. *Under the assumptions of Theorem* **7.1**, *we have*

$$u(x,y,t) = U \overset{x,y}{\widetilde{*}} \left(\frac{\partial^4 f}{\partial x^2 \partial y^2}\right) + V \overset{x,y,t}{\widetilde{*}} \left(\frac{\partial F}{\partial t}\right) + V \overset{x,y}{\widetilde{*}} F(x,y,0).$$

8. Conclusion

In this work, an initial-boundary value problem for the velocity distribution of a viscoelastic flow with generalized fractional Oldroyd-B constitutive model is studied and a Duhamel-type representation of its solution is obtained. The representation is used for the numerical computation of the solution in the particular case of a generalized second grade fluid. Numerical results show that this representation can serve as a basis for an efficient, accurate, and fast numerical technique for solution computation, with the advantage that the solution is calculated in each point independently. The presented technique has the potential for further accuracy improvement and application to multi-dimensional problems.

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References

[1] E. Bazhlekova, Duhamel-type representation of the solutions of nonlocal boundary value problems for the fractional diffusion-wave equation. In: *"Transform Methods and Special Functions, Varna '96" (Proc. 2nd Int. Workshop)*, Bulgarian Academy of Sciences, Sofia (1998), 32–40.

[2] E. Bazhlekova, I. Dimovski, Exact solution of two-term time-fractional Thornley's problem by operational method. *Integral Transform. Spec. Funct.* **25**, No 1 (2014), 61–74.

[3] E. Bazhlekova, B. Jin, R. Lazarov, Z. Zhou, An analysis of the Rayleigh-Stokes problem for a generalized second-grade fluid. *arXiv Preprint*, arXiv:1404.2953 (2014).

[4] C.-M. Chen, F. Liu, V. Anh, A Fourier method and an extrapolation technique for Stokes' first problem for a heated generalized second grade fluid with fractional derivative. *J. Comput. Appl. Math.* **223**, No 2 (2009), 777–789.

[5] C.-M. Chen, F. Liu, V. Anh, Numerical analysis of the Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. *Appl. Math. Comput.* **204**, No 1 (2008), 340–351.
[6] I.H. Dimovski, *Convolutional Calculus*. Kluwer, Dordrecht (1990).
[7] I.H. Dimovski, Y.T. Tsankov, Operational calculi for multidimensional nonlocal evolution boundary value problems. *AIP Conf. Proc.* **1410** (2011), 167–180.
[8] C. Fetecau, M. Jamil, C. Fetecau, D. Vieru, The Rayleigh-Stokes problem for an edge in a generalized Oldroyd-B fluid. *Z. Angew. Math. Phys.* **60**, No 5 (2009), 921–933.
[9] J. Hristov, Integral-balance solution to the Stokes' first problem of a viscoelastic generalized second grade fluid. *Thermal Science*, **16** (2012), 395–410.
[10] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics studies, Elsevier, Amsterdam (2006).
[11] M. Khan, S.H. Ali, T. Hayat, C. Fetecau, MHD flows of a second grade fluid between two side walls perpendicular to a plate through a porous medium. *Int. J. Nonlin. Mech.* **43** (2008), 302–319.
[12] M. Khan, A. Anjum, C. Fetecau, H. Qi, Exact solutions for some oscillating motions of a fractional Burgers' fluid. *Math. Comput. Model.* **51** (2010), 682–692.
[13] M. Khan, A. Anjum, H. Qi, C. Fetecau, On exact solutions for some oscillating motions of a generalized Oldroyd-B fluid. *Z. Angew. Math. Phys.* **61** (2010), 133–145.
[14] Y. Lin, W. Jiang, Numerical method for Stokes' first problem for a heated generalized second grade fluid with fractional derivative. *Numer. Methods Partial Diff. Eq.* **27**, No 6 (2011), 1599–1609.
[15] F. Mainardi, *Fractional Calculus and Waves in Linear Viscoelasticity*. Imperial College Press, London (2010).
[16] F. Mainardi, An historical perspective on fractional calculus in linear viscoelasticity. *Fract. Calc. Appl. Anal.* **15**, No 4 (2012), 712–717; DOI: 10.2478/s13540-012-0048-6; http://link.springer.com/article/10.2478/s13540-012-0048-6.
[17] A. Mohebbi, M. Abbaszadeh, M. Dehghan, Compact finite difference scheme and RBF meshless approach for solving 2D Rayleigh-Stokes problem for a heated generalized second grade fluid with fractional derivatives. *Comput. Methods Appl. Mech. Eng.* **264** (2013), 163–177.
[18] S.P. Näsholm, S. Holm, On a fractional Zener elastic wave equation. *Fract. Calc. Appl. Anal.* **16**, No 1 (2013), 26–50;

DOI: 10.2478/s13540-013–0003-1;
http://link.springer.com/article/10.2478/s13540-013-0003-1.

[19] I. Podlubny, *Fractional Differential Equations.* Academic Press, San Diego, CA (1999).

[20] J. Prüss, *Evolutionary Integral Equations and Applications.* Birkhäuser, Basel - Boston - Berlin (1993).

[21] K. Sakamoto, M. Yamamoto, Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems. *J. Math. Anal. Appl.* **382**, No 1 (2011), 426–447.

[22] Yu. Tsankov, *Operational Calculi for Boundary Value Problems.* PhD thesis, Bulgarian Academy of Sciences, Sofia (2014) (in Bulgarian).

[23] Ch. Wu, Numerical solution for Stokes' first problem for a heated generalized second grade fluid with fractional derivative. *Appl. Numer. Math.* **59**, No 10 (2009), 2571–2583.

[24] C. Zhao, C. Yang, Exact solutions for electro-osmotic flow of viscoelastic fluids in rectangular micro-channels. *Appl. Math. Comput.* **211**, No 2 (2009), 502–509.

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